Hamiltonian formulation of the extended Green-Naghdi equations

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Abstract

A novel method is developed for extending the Green-Naghdi (GN) shallow-water model equation to the general system which incorporates the arbitrary higher-order dispersive effects. As an illustrative example, we derive a model equation which is accurate to the fourth power of the shallowness parameter while preserving the full nonlinearity of the GN equation, and obtain its solitary wave solutions by means of a singular perturbation analysis. We show that the extended GN equations have the same Hamiltonian structure as that of the GN equation. We also demonstrate that Zakharov's Hamiltonian formulation of surface gravity waves is equivalent to that of the extended GN system by rewriting the former system in terms of the momentum density instead of the velocity potential at the free surface.

Keywords: Extended Green-Naghdi equations, Hamiltonian formulation, Water waves

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1. Introduction

The Green-Naghdi (GN) equation which is also known as the Serre or Su-Gardner equations models the fully nonlinear and weakly dispersive surface gravity waves on fluid of finite depth. See Serre [1], Su and Gardner [2] and Green and Naghdi [3]. Although the GN equation approximates the Euler equations for the irrotational flows, it exhibits several remarkable features. In particular, it has a Hamiltonian formulation which provides a unified framework in exploiting the mathematical structure of various model equations such as the Boussinesq, Korteweg-de Vries (KdV) and Camassa-Holm (CH) equations (Camassa and Holm [4] and Camassa *et al* [5]). A large number of works have been devoted to the studies of the GN equation from both analytical and numerical points of view. A review artile by Barthélemy [6] describes the derivation of the GN equation as well as a method for improving the dispersive effect. Furthermore, the improved model equations are tested agianst experiments. The recent article by Bonneton *et al* [7] reviews the high-order numerical methods for the GN equation and the numerical results in comparison with breaking random wave propagation experiments. The following two monographs are concerned with the derivation and mathematical properties of the GN and other water wave equations: Constantin [8] provides an overwiew of some main results and recent developments in nonlinear water waves including breaking waves and tsunamis. Lannes [9] addresses the derivation of various asymptotic model equations and their mathematical analysis which is mainly devoted to the well-posedness of the model equations.

The GN equation incorporates the dispersion of order δ^2, where $\delta = h_0/l$ is the shallowness parameter (h_0: mean depth of the fluid, l: typical length scale of the wave). To improve the dispersion characteristics, various attempts have been made to extend the GN equation. Among them, the model equations have been derived which include the dispersive terms of order δ^4 (Kirby [10], Madson and Schäffer [11, 12] and Gobbi *et al* [13]). Numerical computations have been performed for these equations to examine the wave profiles and the amplitude-velocity relations as well as the effect of dispersion on the wave characteristics. Note, however that whether the proposed higher-order dispersive model equations permit the Hamiltonian formulations has not been discussed so far. In

this paper, we extend the GN equation which is accurate to the dispersive terms of order δ^{2n} while preserving the full nonlinearlity, where n is an arbitrary positive integer. The case $n = 1$ corresponds to the GN equation. We show that the extended model equations have the same Hamiltonian structure as that of the GN equation.

We consider the two-dimensional irrotational flow of an incompressible and inviscid fluid of uniform depth. The effect of surface tension is neglected for the sake of simplicity. The governing equation of the water wave problem is given in terms of the dimensionless variables by

$$\delta^2 \phi_{xx} + \phi_{yy} = 0, \quad -\infty < x < \infty, \quad -1 < y < \epsilon\eta, \tag{1.1}$$

$$\eta_t + \epsilon\phi_x\eta_x = \frac{1}{\delta^2}\phi_y, \quad y = \epsilon\eta, \tag{1.2}$$

$$\phi_t + \frac{\epsilon}{2\delta^2}(\delta^2\phi_x^2 + \phi_y^2) + \eta = 0, \quad y = \epsilon\eta, \tag{1.3}$$

$$\phi_y = 0, \quad y = -1. \tag{1.4}$$

Here, $\phi = \phi(x, y, t)$ is the velocity potential, $\eta = \eta(x, t)$ is the profile of the free surface, and the subscripts x, y, and t appended to ϕ and η denote partial differentiations. The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{x} = lx, \tilde{y} = h_0 y, \tilde{t} = (l/c_0)t, \tilde{\eta} = a\eta$ and $\tilde{\phi} = (gla/c_0)\phi$, where a and c_0 are characteristic scales of the amplitude and velocity of the wave, respectively, and g is the acceleration due to the gravity. Note that $c_0 = \sqrt{gh_0}$ is the long wave phase velocity. In the problem under consideration, one can choose the two independent dimensionless parameters, $\epsilon = a/h_0$ and $\delta = h_0/l$. The former parameter characterizes the magnitude of nonlinearity whereas the latter characterizes the dispersion or shallowness.

In Section 2, we provide a recipe for deriving the model equations. See, for instance Matsuno [14] as for an analogous method which develops a procedure for obtaining the full dispersion model equations of the water wave problem. After completing the construction of the extended GN system, we derive as an example a model equation which is accurate to order δ^4. In Section 3, we show that the extended GN equations can be formulated in a Hamiltonian form by employing an appropriate Lie-Poisson bracket. At the same time, we demonstrate that the extended equations are equivalent to Zakharov's equations

of motion for surface gravity waves. In Section 4, we briefly address the solitary wave solutions of the δ^4 GN model. Finally, Section 5 is devoted to conclusion.

2. Derivation of the extended Green-Naghdi equations

2.1. The extended GN system

We first introduce the mean horizontal velocity component $\bar{u} = \bar{u}(x,t)$ by

$$\bar{u}(x,t) = \frac{1}{h} \int_{-1}^{\epsilon\eta} \phi_x(x,y,t)dy, \quad h = 1 + \epsilon\eta, \tag{2.1}$$

where h is the total depth of the fluid. The horizontal and vertical components of the surface velocity u and v are given respectively by

$$u(x,t) = \phi_x(x,y,t)|_{y=\epsilon\eta}, \tag{2.2}$$

$$v(x,t) = \phi_y(x,y,t)|_{y=\epsilon\eta}. \tag{2.3}$$

Multiplying (2.1) by h and differentiating the resultant expression by x and then using (1.1), (1.4), (2.2) and (2.3), we obtain the relation $(h\bar{u})_x = \epsilon\eta_x u - v/\delta^2$, or since $\epsilon\eta_x = h_x$

$$v = \delta^2\{-(h\bar{u})_x + h_x u\}. \tag{2.4}$$

Substitution of (2.4) into (1.2) yields the evolution equation for h:

$$h_t + \epsilon(h\bar{u})_x = 0. \tag{2.5}$$

An advantage of choosing h and $\bar{u}$ as the dependent variables is that (2.5) becomes an exact equation without any approximation.

Next, we differentiate (2.2) and (2.3) by x and t to obtain the relations

$$u_x = \phi_{xx} + \epsilon\phi_{xy}\eta_x, \quad u_t = \phi_{xt} + \epsilon\phi_{xy}\eta_t, \tag{2.6}$$

$$v_x = \phi_{xy} + \epsilon\phi_{yy}\eta_x, \quad v_t = \phi_{yt} + \epsilon\phi_{yy}\eta_t, \tag{2.7}$$

where the derivatives $\phi_{xx}, \phi_{xy}, \phi_{yy}, \phi_{xt}$ and ϕ_{yt} are evaluated at $y = \epsilon\eta$. Similarly,

$$(\phi_t|_{y=\epsilon\eta})_x = \phi_{xt} + \epsilon\phi_{yt}\eta_x. \tag{2.8}$$

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Eliminating ϕ_{xt} and ϕ_{yt} with use of (2.6) and (2.7), (2.8) becomes

$$(\phi_t|_{y=\epsilon\eta})_x = u_t + v_t h_x - v_x h_t. \tag{2.9}$$

If we differentiate (1.3) by x, insert (2.4) and (2.9) in the resultant expression and use (2.5), we obtain the evolution equation for u:

$$u_t + v_t h_x + \epsilon u u_x + \epsilon h_x u v_x + \eta_x = 0. \tag{2.10}$$

Using (2.5), Eq. (2.10) can be recast into the form

$$[h(u + vh_x)]_t + \epsilon[h(u + vh_x)\bar{u}]_x$$

$$+\epsilon[hv(2h_x\bar{u}_x + h\bar{u}_{xx}) + h(u - \bar{u})(u_x + v_x h_x)] + h\eta_x = 0. \tag{2.11}$$

The system of equations (2.5) and (2.10) (or (2.11)) is equivalent to the basic Euler system (1.1)-(1.4). To obtain the extended GN equations, one needs to express the variables u and v in (2.10) in terms of h and $\bar{u}$. This is always possible as will be exemplified below. Consequently, Eq. (2.10) can be written in the form $\bar{u}_t = \sum_{n=0}^{\infty} \delta^{2n} K_n$, where K_n are polynomials of h and $\bar{u}_{nx}, \bar{u}_{nx,t}, \ (\bar{u}_{nx} = \partial^n \bar{u}/\partial x^n, n = 0, 1, 2...)$. If one retains the terms up to order δ^{2n}, it yields the extended GN equation which is accurate to δ^{2n}. In accordance with this fact, we call the system of equations (2.5) and (2.10) (or (2.11)) with h and $\bar{u}$ being the dependent variables the extended GN system.

2.2. The δ^4 model

Now, we derive the extended GN equation explicitly in the case of $n = 2$ by truncating the system of equations (2.5) and (2.11) at order δ^4, which we call the δ^4 model. To this end, we express the solution of the Laplace equation (1.1) subjected to the boundary condition (1.4) in the form of an infinite series (see, for instance Whitham [15])

$$\phi(x, y, t) = \sum_{n=0}^{\infty} (-1)^n \delta^{2n} \frac{(y + 1)^{2n}}{(2n)!} f_{2nx}, \quad f_{2nx} = \frac{\partial^{2n} f}{\partial x^{2n}}, \tag{2.12}$$

where $f = f(x, t)$ is the velocity potential at the fluid bottom $y = -1$. The expressions (2.1), (2.2) and (2.3) then become

$$\bar{u} = \sum_{n=0}^{\infty} (-1)^n \delta^{2n} \frac{h^{2n}}{(2n + 1)!} f_{(2n+1)x}, \tag{2.13}$$

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$$u = \sum_{n=0}^{\infty} (-1)^n \delta^{2n} \frac{h^{2n}}{(2n)!} f_{(2n+1)x}, \tag{2.14}$$

$$v = \sum_{n=1}^{\infty} (-1)^n \delta^{2n} \frac{h^{2n-1}}{(2n-1)!} f_{2nx}. \tag{2.15}$$

Retaining the terms of order δ^4, (2.13) gives

$$\bar{u} = f_x - \frac{\delta^2}{6} h^2 f_{xxx} + \frac{\delta^4}{120} h^4 f_{xxxxx} + O(\delta^6). \tag{2.16}$$

The inverse relation in which f_x is expressed in terms of h and $\bar{u}$ is achieved by the successive approximation starting from $f_x = \bar{u}$. This leads to

$$f_x = \bar{u} + \frac{\delta^2}{6} h^2 \bar{u}_{xx} + \delta^4 \left\{ \frac{h^2}{36} (h^2 \bar{u}_{xx})_{xx} - \frac{h^4}{120} \bar{u}_{xxxx} \right\} + O(\delta^6). \tag{2.17}$$

On the other hand, u from (2.14) is expanded as

$$u = f_x - \frac{\delta^2}{2} h^2 f_{xxx} + \frac{\delta^4}{24} h^4 f_{xxxxx} + O(\delta^6). \tag{2.18}$$

The last step is to introduce f_x from (2.17) into (2.18), giving rise to the relation

$$u = \bar{u} - \frac{\delta^2}{3} h^2 \bar{u}_{xx} - \delta^4 \left\{ \frac{1}{45} h^4 \bar{u}_{xxxx} + \frac{2}{9} h^3 h_x \bar{u}_{xxx} + \frac{1}{18} h^2 (h^2)_{xx} \bar{u}_{xx} \right\} + O(\delta^6). \tag{2.19}$$

In view of (2.19), the expression of v from (2.4) can be written simply as

$$v = -\delta^2 h \bar{u}_x - \frac{\delta^4}{3} h^2 h_x \bar{u}_{xx} + O(\delta^6). \tag{2.20}$$

The evolution equation for $\bar{u}$ follows by substituting (2.19) and (2.20) into (2.10) and using (2.5) to replace h_t. After some manipulations, we arrive at the compact equation for $\bar{u}$:

$$\bar{u}_t + \epsilon \bar{u} \bar{u}_x + \eta_x = \frac{\delta^2}{3h} \left\{ h^3 (\bar{u}_{xt} + \epsilon \bar{u} \bar{u}_{xx} - \epsilon \bar{u}_x^2) \right\}_x$$

$$+ \frac{\delta^4}{45h} \left[\left\{ h^5 (\bar{u}_{xxt} + \epsilon \bar{u} \bar{u}_{xxx} - 5\epsilon \bar{u}_x \bar{u}_{xx}) \right\}_x - 3\epsilon h^5 \bar{u}_{xx}^2 \right]_x + O(\delta^6). \tag{2.21}$$

If we multiply (2.21) by h and use (2.5), then we can put it in the conservation form

$$(h\bar{u})_t + \epsilon \left(h\bar{u}^2 + \frac{h^2}{2\epsilon^2} \right)_x = \frac{\delta^2}{3} \left\{ h^3 (\bar{u}_{xt} + \epsilon \bar{u} \bar{u}_{xx} - \epsilon \bar{u}_x^2) \right\}_x$$

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$$+\frac{\delta^4}{45}\left[\left\{h^5(\bar{u}_{xxt} + \epsilon\bar{u}\bar{u}_{xxx} - 5\epsilon\bar{u}_x\bar{u}_{xx})\right\}_x - 3\epsilon h^5\bar{u}_{xx}^2\right]_x + O(\delta^6). \tag{2.22}$$

The system of equations (2.5) and (2.21) (or (2.22)) is the extended version of the GN equation which is accurate to order δ^4 and which retains the full nonlinearity. Recall that the amplitude parameter ϵ which characterizes the magnitude of nonlinearity enters into h through the relation $h = 1 + \epsilon\eta$. What is meant by "full nonlinearity" is that the parameter ϵ is assumed to be of order 1.

If we use the approximation $h^n = 1 + \epsilon n\eta + (\epsilon^2)$ in (2.21) under the assumption $\epsilon \ll 1$, where n is a positive integer and ignore the $\epsilon\delta^2$ and higher-order terms, it reduces to the classical Boussinesq system (Whitham [15])

$$h_t + \epsilon(h\bar{u})_x = 0, \quad \bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3}\bar{u}_{xxt}, \tag{2.23}$$

and to the original GN system (Serre [1], Su and Gardner [2] and Green and Naghdi [3])

$$h_t + \epsilon(h\bar{u})_x = 0, \quad \bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \epsilon\bar{u}_x^2)\right\}_x, \tag{2.24}$$

if we ignore the δ^4 terms while preserving the full nonlinearity.

Remark 1. Equation (2.21) involves the nonlinear terms of order ϵ^5. If one retains the $\epsilon\delta^2, \epsilon\delta^4$ and $\epsilon^2\delta^2$ terms, then it reduces to a model equation presented in Madson and Schäffer [11] (see Eq. (3.11) therein). If one imposes a relation between the parameters ϵ and δ, then one can derive the higher-order versions of the various shallow-water model equations. For instance, the scaling $\epsilon = O(\delta^2) \ll 1$ will lead to the higher-order KdV equation and to the higher-order CH equation under the scaling $\epsilon = O(\delta) \ll 1$. See Johnson [16] and Constantin and Lannes [17].

Remark 2. The linearization of the system of equations (2.5) and (2.21) about the uniform state $h = 1$ and $\bar{u} = 0$ yields the system of linear equations for η and $\bar{u}$

$$\eta_t + \bar{u}_x = 0, \quad \bar{u}_t + \eta_x = \frac{\delta^2}{3}\bar{u}_{xxt} + \frac{\delta^4}{45}\bar{u}_{xxxxt}. \tag{2.25}$$

Eliminating the variable η from this system, one obtains the linear wave equation for $\bar{u}$

$$\bar{u}_{tt} - \bar{u}_{xx} = \frac{\delta^2}{3}\bar{u}_{xxtt} + \frac{\delta^4}{45}\bar{u}_{xxxxtt}. \tag{2.26}$$

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The dispersion relation for equation (2.26) is given by

$$\omega^2 = \frac{k^2}{1 + \frac{(\delta k)^2}{3} - \frac{(\delta k)^4}{45}}. \tag{2.27}$$

For the wave propagating to the right, the phase velocity $c_p = \omega/k$ and the group velocity $v_g = d\omega/dk$ have the series expansions

$$c_p = 1 - \frac{1}{6}(\delta k)^2 + \frac{19}{360}(\delta k)^4 - \frac{37}{2160}(\delta k)^6 + O((\delta k)^8), \tag{2.28}$$

$$c_g = 1 - \frac{1}{2}(\delta k)^2 + \frac{19}{72}(\delta k)^4 - \frac{259}{2160}(\delta k)^6 + O((\delta k)^8). \tag{2.29}$$

These expressions are compared with the exact dispersion relation $\omega^2 = k \tanh k\delta/\delta$, as well as the corresponding phase and group velocities which are given respectively by

$$c_p = 1 - \frac{1}{6}(\delta k)^2 + \frac{19}{360}(\delta k)^4 - \frac{55}{3024}(\delta k)^6 + O((\delta k)^8), \tag{2.30}$$

$$c_g = 1 - \frac{1}{2}(\delta k)^2 + \frac{19}{72}(\delta k)^4 - \frac{55}{432}(\delta k)^6 + O((\delta k)^8). \tag{2.31}$$

As expected, both velocities coincide up to order $(k\delta)^4$. Slight differences appear at order $(k\delta)^6$.

Remark 3. The system of equations (2.5) and (2.22) exhibits the following three conservation laws:

$$M = \int_{-\infty}^{\infty} (h - 1)dx, \tag{2.32}$$

$$P = \int_{-\infty}^{\infty} h\bar{u}\, dx, \tag{2.33}$$

$$H = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \left[h\bar{u}^2 + \frac{\delta^2}{3}h^3\bar{u}_x^2 - \frac{\delta^4}{45}h^5\bar{u}_{xx}^2 + \frac{1}{\epsilon^2}(h-1)^2 \right] dx, \tag{2.34}$$

which represent the conservation of the mass, momentum and total energy, respectively. The first two conservation laws follow directly from (2.5) and (2.22) whereas the last one can be checked by means of a straightforward calculation using (2.5) and (2.21). A factor ϵ^2 attached in front of the integral sign in (2.34) is only for convenience.

3. Hamiltonian formulation

3.1. Hamiltonian structure

Here, we show that the extended GN system (2.5) and (2.11) can be formulated as a Hamiltonian system by introducing an appropriate Lie-Poisson bracket. As pointed out by Zakharov [18], the basic Euler system of equations (1.1)-(1.4) conserves the total energy (i.e., kinetic plus potential energies)

$$H = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \left[\int_{-1}^{\epsilon\eta} \left(\phi_x^2 + \frac{1}{\delta^2}\phi_y^2 \right) dy + \eta^2 \right] dx. \tag{3.1}$$

This fact can be confirmed easily using (1.1)-(1.4). We substitute (2.12) into (3.1) and then perform the integration with respect to y to express H in terms of $f_x, f_{xx}, ...$ and h. Eliminating $f_x, f_{xx}, ...$ with the aid of the higher-order version of (2.17) yields a series expansion of H in δ^2 as

$$H = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} f_n(h, h_x, h_{xx}, ...; \delta^2) \bar{u}_{nx}^2 + \eta^2 \right] dx = \epsilon^2 \sum_{n=0}^{\infty} \delta^{2n} H_n, \tag{3.2}$$

where $f_0 = h$ and $f_n (\geq 1)$ are polynomials of $h, h_x, h_{xx}, ...$ (δ^2 is simply a parameter). Specifically, the first four H_n read

$$H_0 = \frac{1}{2} \int_{-\infty}^{\infty} \left[h\bar{u}^2 + \frac{1}{\epsilon^2}(h-1)^2 \right] dx, \quad H_1 = \frac{1}{6} \int_{-\infty}^{\infty} h^3 \bar{u}_x^2 dx, \quad H_2 = -\frac{1}{90} \int_{-\infty}^{\infty} h^5 \bar{u}_{xx}^2 dx,$$

$$H_3 = \frac{1}{1890} \int_{-\infty}^{\infty} \left[2h^7 \bar{u}_{xxx}^2 - 7h^5(hh_x)_x \bar{u}_{xx}^2 \right] dx. \tag{3.3}$$

Let us now introduce the momentum density $m = m(x,t)$ by

$$\epsilon m = \frac{\delta H}{\delta \bar{u}}, \tag{3.4}$$

where the operator $\delta/\delta\bar{u}$ is the variational derivative defined by the relation

$$\frac{\partial}{\partial \epsilon} H[\bar{u} + \epsilon w]|_{\epsilon=0} = \int_{-\infty}^{\infty} \frac{\delta H}{\delta \bar{u}} w\, dx, \tag{3.5}$$

for arbitrary function w. In terms of m above, the Hamiltonian (or total energy) (3.2) can be put into the simple form

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left[\epsilon \bar{u} m + (h-1)^2 \right] dx. \tag{3.6}$$

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Actually, it follows from (3.2) and (3.4) that

$$m = \epsilon \sum_{n=0}^{\infty} (-1)^n (f_n \bar{u}_{nx})_{nx}. \tag{3.7}$$

Multiplying (3.7) by $\bar{u}$ and integrating by parts, we obtain

$$\epsilon \int_{-\infty}^{\infty} \bar{u}m \, dx = \epsilon^2 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} f_n(h, h_x, h_{xx}, ...) \bar{u}_{nx}^2 \, dx, \tag{3.8}$$

which, substituted in (3.2), gives (3.6).

To proceed, we regard m and h as the dependent variables instead of $\bar{u}$ and h. Then, the variation of H in the former variables is found to be

$$\delta H = \int_{-\infty}^{\infty} \left[\epsilon \bar{u} \, \delta m - \frac{\epsilon^2}{2} \left\{ \sum_{n=0}^{\infty} \sum_{s=0}^{p(n)} (-1)^s \left(\frac{\partial f_n}{\partial h_{sx}} \bar{u}_{nx}^2 \right)_{sx} - \frac{2}{\epsilon^2} (h-1) \right\} \delta h \right] dx, \tag{3.9}$$

where $p(n)$ is a nonnegative positive integer depending on n. In particular, $p(0) = p(1) = 0$. To verify (3.9), we modify H from (3.6) as

$$H = \int_{-\infty}^{\infty} \left[\epsilon \bar{u}m - \frac{\epsilon}{2} \bar{u}m + \frac{1}{2}(h-1)^2 \right] dx, \tag{3.10}$$

and insert the expression of m from (3.7) into the middle term in the integrand, and then take the variations with respect to $m, \bar{u}$ and h. The coefficient of $\delta \bar{u}$ vanishes by virtue of (3.7), resulting in (3.9). It readily follows from (3.9) that

$$\frac{\delta H}{\delta m} = \epsilon \bar{u}, \tag{3.11}$$

$$\frac{\delta H}{\delta h} = -\frac{\epsilon^2}{2} \sum_{n=0}^{\infty} \sum_{s=0}^{p(n)} (-1)^s \left(\frac{\partial f_n}{\partial h_{sx}} \bar{u}_{nx}^2 \right)_{sx} + h - 1. \tag{3.12}$$

Now, our main result is given by the following theorem.

Theorem 1. *The extended GN system (2.5) and (2.11) can be written in the following Hamiltonian form*

$$\begin{pmatrix} m_t \\ h_t \end{pmatrix} = - \begin{pmatrix} \partial_x m + m \partial_x & h \partial_x \\ \partial_x h & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta m \\ \delta H/\delta h \end{pmatrix}, \tag{3.13}$$

where the derivative $\partial_x \equiv \partial/\partial x$ operates on all terms it multiplies to the right. Introduce the Lie-Poisson bracket between any pair of smooth functionals F and G

$$\{F, G\} = -\int_{-\infty}^{\infty} \left[\frac{\delta F}{\delta m}(m\partial_x + \partial_x m)\frac{\delta G}{\delta m} + \frac{\delta F}{\delta m} h\partial_x \frac{\delta G}{\delta h} + \frac{\delta F}{\delta h} \partial_x h \frac{\delta G}{\delta m} \right] dx. \tag{3.14}$$

Then, the equations of motion for m and h from (3.13) are expressed as

$$m_t = \{m, H\}, \quad h_t = \{h, H\}. \tag{3.15}$$

Proof. The proof of Theorem 1 relies on the following two relations:

$$m = \epsilon h(u + vh_x), \tag{3.16}$$

$$m\bar{u}_x + \frac{h}{\epsilon}\left(\frac{\delta H}{\delta h}\right)_x = \epsilon[hv(2h_x\bar{u}_x + h\bar{u}_{xx}) + h(u - \bar{u})(u_x + v_x h_x)] + h\eta_x, \tag{3.17}$$

which we shall address first and then proceed to the proof of Theorem 1.

We modify the Hamiltonian H from (3.1). First, use (1.1) to obtain

$$\phi_x^2 + \frac{\phi_y^2}{\delta^2} = (\phi_x\phi)_x + \frac{1}{\delta^2}(\phi_y\phi)_y.$$

If we introduce this expression into (3.1) and perform the integration with respect to y under the boundary condition (1.4) as well as (2.2) and (2.3), we deduce

$$H = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \left[\left(-\epsilon\eta_x u + \frac{v}{\delta^2}\right)\psi + \eta^2 \right] dx,$$

where ψ is the velocity potential at the free surface given by

$$\psi = \phi(x, \epsilon\eta, t). \tag{3.18}$$

By virtue of (2.4), the above Hamiltonian reduces, after integrating by parts, to

$$H = \frac{\epsilon^2}{2} \int_{-\infty}^{\infty} \left[h\bar{u}\psi_x + \eta^2 \right] dx.$$

Equating this expression with (3.6), we arrive at the integral identity

$$\int_{-\infty}^{\infty} \bar{u}(m - \epsilon h\psi_x)dx = 0.$$

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Since this must hold for arbitrary $\bar{u}$, we obtain an important relation

$$m = \epsilon h \psi_x, \tag{3.19}$$

which connects the variable ψ_x with the variable m. It follows by differentiating (3.18) by x and using (2.2) and (2.3) that

$$\psi_x = \phi_x(x, \epsilon\eta, t) + \epsilon\phi_y(x, \epsilon\eta, t)\eta_x = u + vh_x,$$

which, coupled with (3.19), yields (3.16).

We substitute m from (3.16) into (3.17), divide the resultant expression by h and use (2.4). The relation thus obtained can be integrated by x to give

$$\frac{1}{\epsilon^2}\frac{\delta H}{\delta h} = \frac{1}{2}u^2 + \frac{v^2}{2\delta^2} + hv\bar{u}_x - u\bar{u} + \frac{1}{\epsilon^2}(h-1). \tag{3.20}$$

By taking the variational derivative of H from (3.1) with respect to h, we obtain

$$\frac{\delta H}{\delta h} = \frac{\epsilon^2}{2}\left(u^2 + \frac{v^2}{\delta^2} + \frac{2\eta}{\epsilon}\right) + \epsilon^2 \int_{-\infty}^{\infty}\left(-\epsilon\eta_x u + \frac{v}{\delta^2}\right)\frac{\delta\phi}{\delta h}\bigg|_{y=\epsilon\eta} dx.$$

The following relation comes from the formula $\delta h(x,t)/\delta h(x',t) = \delta(x-x')$, where $\delta(x-x')$ is Dirac's delta function:

$$\frac{\delta\phi(x,y,t)}{\delta h(x',t)}\bigg|_{y=\epsilon\eta} = \frac{\delta\psi(x,t)}{\delta h(x',t)} - v\delta(x-x').$$

We use this relation and (2.4) in the expression of $\delta H/\delta h$ to obtain

$$\frac{\delta H}{\delta h} = \frac{\epsilon^2}{2}\left(u^2 + \frac{v^2}{\delta^2} + \frac{2\eta}{\epsilon}\right) + \epsilon^2(h\bar{u})_x v + \epsilon^2 \int_{-\infty}^{\infty} h\bar{u}\frac{\delta\psi_x}{\delta h}dx.$$

Last, substituting the above expression into the left-hand side of (3.20), the relation to be proved reduces to

$$(h\bar{u})_x v + \int_{-\infty}^{\infty} h\bar{u}\frac{\delta\psi_x}{\delta h}dx = hv\bar{u}_x - u\bar{u}. \tag{3.21}$$

The integral on the left-hand side of (3.21) can be evaluated by using the relation $\psi_x = m/\epsilon h$ from (3.19), which leads to

$$\int_{-\infty}^{\infty} h\bar{u}\frac{\delta\psi_x}{\delta h}dx = -\frac{m\bar{u}}{\epsilon h} = -(u + vh_x)\bar{u},$$

showing that (3.21) holds identically. Hence, we establish (3.17).

Now, Theorem 1 follows immediately from (3.16) and (3.17). First, we note that the introduction of (3.11) into (3.13) yields the system of evolution equations for m and h:

$$m_t = -\epsilon(\bar{u}m)_x - \epsilon m\bar{u}_x - h\left(\frac{\delta H}{\delta h}\right)_x, \qquad (3.22)$$

$$h_t = -\epsilon(h\bar{u})_x. \qquad (3.23)$$

Equation (3.23) is just Eq. (2.5). If we substitute (3.16) and (3.17) into (3.22), we see that the resultant equation is in agreement with Eq. (2.11). It is obvious that (3.15) with (3.14) is equivalent to (3.13). $\square$

Remark 4. The bracket (3.14) has been used in Holm [19] to verify the Hamiltonian structure of two-dimensional shallow-water hydrodynamics with nonlinear dispersion, as well as that of the GN equation which has been detailed by Constantin [20]. It has a skew-symmetry $\{F, G\} = -\{G, F\}$ and satisfies the Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ (Constantin [20]).

Remark 5. If we retain the terms of order δ^4, then m and $\delta H/\delta h$ are found to be

$$m = \epsilon\left[h\bar{u} - \frac{\delta^2}{3}(h^3\bar{u}_x)_x - \frac{\delta^4}{45}(h^5\bar{u}_{xx})_{xx}\right], \qquad (3.24)$$

$$\frac{\delta H}{\delta h} = -\frac{\epsilon^2}{2}\left\{\bar{u}^2 + \delta^2 h^2\bar{u}_x^2 - \frac{\delta^4}{9}h^4\bar{u}_{xx}^2 - \frac{2}{\epsilon^2}(h-1)\right\}. \qquad (3.25)$$

Equation (3.22) then becomes

$$m_t = -\epsilon\bar{u}m_x - 2\epsilon\bar{u}_x m + \frac{\epsilon^2}{2}h\left[\bar{u}^2 + \delta^2 h^2\bar{u}_x^2 - \frac{\delta^4}{9}h^4\bar{u}_{xx}^2 - \frac{2}{\epsilon^2}(h-1)\right]_x. \qquad (3.26)$$

One can check by a direct computation that (3.26) indeed reproduces (2.22).

Remark 6. In addition to the conservation of mass, momentum and total energy which are given respectively by (2.32), (2.33) and (2.34), the extended GN system exhibits the fourth conservation law $L \equiv \int_{-\infty}^{\infty} m/h\, dx$. This immediately follows from the equation $(m/h)_t = -(\epsilon\bar{u}m/h + \delta H/\delta h)_x$ which is derived from the system of equations (3.22) and (3.23). In view of (3.16), the density of L is written as $m/h = \epsilon(u + vh_x)$. Since the

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unit tangent vector $\mathbf{t}$ to the free surface is given by $\mathbf{t} = (1/\sqrt{1+h_x^2}, h_x/\sqrt{1+h_x^2})$ and the surface velocity by $\mathbf{u} = (u, v)$, the density m/h is equal to the tangential component $u_t \equiv \mathbf{u} \cdot \mathbf{t}$ of the surface velocity multiplied by a factor $\epsilon\sqrt{1+h_x^2}$. It turns out that L is represented by the line integral of the tangent velocity along the free surface. To be more specific, $L = \epsilon \int_C u_t ds$, where $ds = \sqrt{1+h_x^2}\,dx$ is the line element of the curve C representing the free surface. Using (3.19), this integral can be evaluated explicitly as $L = \epsilon(\psi_+ - \psi_-)$ with $\psi_\pm \equiv \lim_{x\to\pm\infty}\psi(x,t)$, showing that the conserved quantity L is determined only by the limiting values $\psi_\pm$ of the velocity potential at the free surface. Note also that the quantity $\int_{-\infty}^{\infty} m\,dx$ is conserved. However, it is not independent of the momentum P given by (2.33). Actually, integrating the relation (3.7) with $f_0 = h$, the above conserved quantity reduces to $\epsilon \int_{-\infty}^{\infty} h\bar{u}\,dx = \epsilon P$.

3.2. Relation to Zakharov's Hamiltonian formulation

The water wave problem posed by (1.1)-(1.4) permits a Hamiltonian formulation. Following Zakharov [18], the equations of motion for surface gravity waves can be written in terms of the variables h and ψ_x as

$$h_t = -\frac{1}{\epsilon}\left(\frac{\delta H}{\delta \psi_x}\right)_x, \tag{3.27}$$

$$\psi_{xt} = -\frac{1}{\epsilon}\left(\frac{\delta H}{\delta h}\right)_x. \tag{3.28}$$

Note that the variable ψ_x is used instead of ψ. This choice is suitable for the present analysis. If we define the bracket between any pair of smooth functionals F and G by

$$\{F, G\} = -\frac{1}{\epsilon}\int_{-\infty}^{\infty}\left[\frac{\delta F}{\delta h}\left(\frac{\delta G}{\delta \psi_x}\right)_x - \left(\frac{\delta F}{\delta \psi_x}\right)_x \frac{\delta G}{\delta h}\right]dx, \tag{3.29}$$

then, Eqs. (3.27) and (3.28) are expressed as

$$h_t = \{h, H\}, \quad \psi_{xt} = \{\psi_x, H\}. \tag{3.30}$$

Under the above setting, the following theorem is established:

Theorem 2. *The system of equations (3.13) is equivalent to Zakharov's system of equations (3.27) and (3.28).*

Proof. First, we change the variables h and ψ_x to h and m and rewrite Eqs. (3.27) and (3.28) in terms of the latter variables. To this end, we evaluate the variation of the Hamiltonian H in two alternative ways:

$$\int_{-\infty}^{\infty} \left[\frac{\delta H}{\delta h}\bigg|_{\psi_x:\text{fixed}} \delta h + \frac{\delta H}{\delta \psi_x}\bigg|_{h:\text{fixed}} \delta \psi_x \right] dx = \int_{-\infty}^{\infty} \left[\frac{\delta H}{\delta h}\bigg|_{m:\text{fixed}} \delta h + \frac{\delta H}{\delta m}\bigg|_{h:\text{fixed}} \delta m \right] dx. \tag{3.31}$$

We compute the variation of m from (3.19) as $\delta m = \epsilon \psi_x \delta h + \epsilon h \delta \psi_x$ and subsitute this expression into the right-hand side of (3.31) and then compare the coefficient of δh and $\delta \psi_x$ on both sides. It follows that

$$\frac{\delta H}{\delta h}\bigg|_{\psi_x:\text{fixed}} = \frac{\delta H}{\delta h}\bigg|_{m:\text{fixed}} + \epsilon \frac{\delta H}{\delta m}\bigg|_{h:\text{fixed}} \psi_x, \tag{3.32}$$

$$\frac{\delta H}{\delta \psi_x}\bigg|_{h:\text{fixed}} = \epsilon h \frac{\delta H}{\delta m}\bigg|_{h:\text{fixed}}. \tag{3.33}$$

Taking into account (3.11) and (3.19), the above relations become

$$\frac{\delta H}{\delta h}\bigg|_{\psi_x:\text{fixed}} = \frac{\delta H}{\delta h}\bigg|_{m:\text{fixed}} + \frac{\epsilon \bar{u} m}{h}, \tag{3.34}$$

$$\frac{\delta H}{\delta \psi_x}\bigg|_{h:\text{fixed}} = \epsilon^2 h \bar{u}. \tag{3.35}$$

These formulas enable us to transform the Zakharov system to the extended GN system. Actually, Eq. (3.27) with (3.35) yields Eq. (3.23). On the other hand, introducing (3.19) and (3.34) into (3.28), we obtain

$$\left(\frac{m}{h} \right)_t = \frac{m_t}{h} - \frac{m h_t}{h^2} = -\left(\frac{\delta H}{\delta h} \right)_x - \epsilon \left(\frac{\bar{u} m}{h} \right)_x. \tag{3.36}$$

where we have used the notation $(\delta H/\delta h)|_{m:\text{fixed}} = \delta H/\delta h$ for simplicity. If we replace h_t in the middle expression of (3.36) by the right-hand side of Eq. (3.23), then the equation reduces to Eq. (3.22) after multiplying h on both sides. Last, we use (3.32) and (3.33) with H being replaced by F and G, respectively and (3.19) for ψ_x to rewrite the bracket (3.29) in terms of h and m. The resultant expression becomes

$$\{F, G\} = -\int_{-\infty}^{\infty} \left[\left(\frac{\delta F}{\delta h} + \frac{m}{h} \frac{\delta F}{\delta m} \right) \left(h \frac{\delta G}{\delta m} \right)_x - \left(h \frac{\delta F}{\delta m} \right)_x \left(\frac{\delta G}{\delta h} + \frac{m}{h} \frac{\delta G}{\delta m} \right) \right] dx. \tag{3.37}$$

Integrating the second term of (3.37) by parts and rearranging the integrand, we confirm that (3.37) transforms to the bracket defined by (3.14).   □

4. Solitary wave solutions

We seek the solitary wave solutions of the extended GN system (2.5) and (2.22) in the form of the traveling wave $h = h(\xi), \bar{u} = \bar{u}(\xi), (\xi = x - ct)$. We impose the boundary conditions $h \to 1, h' \to 0$ and $\bar{u}, \bar{u}', \bar{u}'', \bar{u}''', \bar{u}'''' \to 0$ as $|x| \to \infty$, where $c(> 0)$ is the velocity of the solitary wave and primes refer to differentiation with respect to ξ. If we introduce these forms into (2.5) and (2.22) and integrate each equation once with respect to ξ while taking into account the boundary conditions, we deduce

$$-c(h - 1) + \epsilon h\bar{u} = 0, \tag{4.1}$$

$$-ch\bar{u} + \epsilon\left(h\bar{u}^2 + \frac{h^2 - 1}{2\epsilon^2}\right) = \frac{\delta^2}{3}h^3(-c\bar{u}'' + \epsilon\bar{u}\bar{u}'' - \epsilon(\bar{u}')^2)$$

$$+\frac{\delta^4}{45}\left[\{h^5(-c\bar{u}''' + \epsilon\bar{u}\bar{u}''' - 5\epsilon\bar{u}'\bar{u}'')\}' - 3\epsilon h^5(\bar{u}'')^2\right]. \tag{4.2}$$

We use (4.1) to express $\bar{u}$ in terms of h as

$$\bar{u} = \frac{c}{\epsilon}\left(1 - \frac{1}{h}\right). \tag{4.3}$$

Substituting (4.3) into (4.2), multiplying the resultant equation by h'/h^2 and then integrating, we obtain the third-order nonlinear differential equation for h:

$$h^3 - (c^2 + 2)h^2 + (2c^2 + 1)h - c^2$$

$$= -\frac{\delta^2 c^2}{3}(h')^2 - \frac{\delta^4 c^2}{45}\left[2h^2h'h''' - h^2(h'')^2 + 2h(h')^2h'' - 12(h')^4\right]. \tag{4.4}$$

If we neglect the δ^4 terms on the right-hand side of (4.4), then it reduces to the corresponding equation for the GN equation. Unlike the Boussinesq system (2.23), it exhibits the analytical solitary wave solution of the following form which has been obtained for the first time by Serre [1] and later by Su and Gardner [2]:

$$h = 1 + (c^2 - 1)\operatorname{sech}^2\frac{\sqrt{3(c^2 - 1)}}{2c\delta}\xi, \quad (c > 1). \tag{4.5}$$

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An inspection reveals, however that Eq. (4.4) would not have analytical solutions and hence numerical analysis may be necessary to extract the characteristics associated with the solitary waves such as the surface profile and the velocity-amplitude relation. Instead, we perform a singular perturbation analysis.

Note first that the parameter δ in Eq. (4.4) disappears by means of the scaling $\xi \to \delta\xi$. Furthermore, the form of the solitary wave solution (4.5) suggests that an appropriate scaling of the variable ξ would be $b\zeta$, where $\zeta = \xi/\delta$ and the unknown parameter b plays the role of the wavenumber which is determined in the course of the pertubation analysis. In accordance with this observation, we expand h and c^2 in powers of ϵ as

$$h(b\zeta) = 1 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + O(\epsilon^4), \tag{4.6}$$

$$c^2 = 1 + \epsilon c_1 + \epsilon^2 c_2 + \epsilon^3 c_3 + O(\epsilon^4), \tag{4.7}$$

with

$$b^2 = \epsilon b_1 + \epsilon^2 b_2 + \epsilon^3 b_3 + O(\epsilon^4). \tag{4.8}$$

We employ the conditions $h_1(0) = 1$ and $h_j(0) = 0, j \geq 2$ which define the amplitude ϵ by $h(0) = 1 + \epsilon$. The method for constructing solitary wave solutions is well-known. See, for instance Fenton [21]. Hence, we summarize the result.

The solution to the boundary value problem includes the secular term like $\zeta \operatorname{sech}^2 b\zeta \tanh b\zeta$. To ensure the uniform validity of the expansion (4.6), one must eliminate the secular terms by choosing the coefficients b_j and c_j $(j = 1, 2, ...)$ appropriately. The solitary wave solution thus constructed can be written as

$$h(b\zeta) = 1 + \epsilon \operatorname{sech}^2 b\zeta - \frac{3}{4}\epsilon^2 \operatorname{sech}^2 b\zeta \tanh^2 b\zeta + \frac{5}{8}\epsilon^3 \left(1 - \frac{149}{50}\operatorname{sech}^2 b\zeta\right) \operatorname{sech}^2 b\zeta \tanh^2 b\zeta + O(\epsilon^4), \tag{4.9}$$

$$c^2 = 1 + \epsilon - \frac{1}{20}\epsilon^2 - \frac{1}{10}\epsilon^3 + O(\epsilon^4), \tag{4.10}$$

$$b = \sqrt{\frac{3}{4}\epsilon}\left(1 - \frac{5}{8}\epsilon + \frac{71}{128}\epsilon^2 + O(\epsilon^3)\right). \tag{4.11}$$

The above solution coincides with the third-order solution obtained in Fenton [21] and Grimshaw [22] up to order ϵ^2. The discrepancy appears at order ϵ^3. The reason is obvious since the latter takes into account the δ^6 terms which have been neglected in our model

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equation. To be more specific, the scaling $\epsilon = O(\delta^2)$ is assumed in Fenton [21] and Grimshaw [22] so that the δ^6 terms turn out to be comparable with the ϵ^3 terms.

Remark 7. Introduce the quantity Q by

$$Q = \int_{-\infty}^{\infty} m \left(1 - \frac{1}{h}\right) dx, \tag{4.12}$$

which has been shown to be conserved by the extended GN system (See Remark 6). For the δ^4 model, the solitary wave solution $h(\xi)$ and $m(\xi)$ is a critical point of the functional $H - cQ$ with respect to m and h. Specifically,

$$\frac{\delta H}{\delta m} - c \frac{\delta Q}{\delta m} = 0, \quad \frac{\delta H}{\delta h} - c \frac{\delta Q}{\delta h} = 0. \tag{4.13}$$

This fact can be confirmed by a direct computation using (3.11), (3.24), (3.25) and (4.4). Thus, a variational characterization is possible for the solitary wave solution of the δ^4 model, even if it does not exhaibit exact solitary wave solutions. It will be an important issue to establish this assertion for the general δ^{2n} $(n \geq 3)$ model. In the case of the GN equation, the similar result has been pointed out and used to prove the linear stability of the solitary wave solution (4.5) (Li [23]).

5. Conclusion

We have presented a novel method for extending the GN equation to the general system which incorporates the arbitrary higher-order dispersive terms while preserving the full nonlinearity. As an illustrative example, we have derived a model equation which is accurate to order δ^4 for which the perturbation analysis reproduces the known solitary wave solutions at the level of order ϵ^2. Nevertheless, the numerical analysis will be necessary to extract the fully nonlinear nature of the model equation, as has been performed for the GN equation by Li *et al* [24] and Mitsotakis *et al* [25]. We have shown that our extended system permits the same Hamiltonian structure as that of the GN equation. In the process of establishing this fact, the energy integral of the basic Euler system has been used efficiently. We have also verified that Zakharov's Hamiltonian formulation of the current water wave problem is equivalent to that of the extended GN equations. This has been accomplished by rewriting Zakharov's equations in terms of the total depth of

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the fluid and momentum density. The extension of our model equations to the more general setting with an uneven bottom topography and its three-dimensional generalization will be done along with the same procedure as that developed here. In any case, there are many things to be resolved by future study.

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